AD-A102 057 MARYLAND UNIV COLLEGE PARK COMPUTER VISION LAB F/6 12/1
ON AN ESTIMATION SCHEME FOR GAUSS MARKOV RANDOM FIELD MODELS.(U)
APR 81 R CHELLAPPA AFOSR-77-3271
UNCLASSIFIED TR-1040 AFOSR-TR-81-0556 NL

END
DATE
FILMED
8 81
DTIC

AFOSR-TR- 81 -0556 LEVEL II

10

AD A102057

COMPUTER SCIENCE TECHNICAL REPORT SERIES

DTIC ELECTE JUL 28 1981 S D

D

UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND
20742

DTIC FILE COPY

Approved for public release; distribution unlimited.

81 7 24 045

LEVEL II

(10)

TR-1040
AFOSR-77-3271

April, 1981

ON AN ESTIMATION SCHEME FOR GAUSS MARKOV RANDOM FIELD MODELS

R. Chellappa

Computer Vision Laboratory
Computer Science Center
University of Maryland
College Park, Maryland

ABSTRACT

In an earlier report [1] a consistent estimation scheme was given for Gaussian Markov random field models. In this report we consider some statistical properties of the resulting estimate. Specifically, we derive an expression for the asymptotic mean square error of the estimate for a general model and compare the efficiency of this estimate with the popular coding estimate for a simple first order isotropic model.

The support of the U.S. Air Force Office of Scientific Research under Grant AFOSR-77-3271 is gratefully acknowledged, as is the help of Sally Atkinson in preparing this paper. The author is indebted to Prof. R. L. Kashyap and Prof. A. Rosenfeld for helpful discussions.

1. Introduction

Markov random field (MRF) models are of much interest in image analysis and processing. For instance, they have been used in texture analysis [2,3] and image restoration [4-6]. They are also of interest in the analysis of field data [7]. MRF models characterize the special nature of the statistical dependency of intensity levels over a neighborhood in an image. If $y(s)$ denotes the intensity level at location s, then $y(s)$ is written as a linear combination of intensity levels $\{y(s+r), r \in N\}$ (where N is known as the neighbor set of dependence) and additive noise. The members of set N are pairs of integers (k,ℓ) not including $(0,0)$. For example, the first order MRF model results when $N=\{(0,1),(1,0),(0,-1),(-1,0)\}$ and the second order MRF model results when $N=\{(-1,0),(1,0),(0,-1),(0,1),(-1,-1),(-1,1),(1,-1),(1,1)\}$. Each MRF model is characterized by a set of linear weights and the variance of the additive noise.

Suppose we are given an array of intensity level variations $\{y(s), s \in \Omega\}$, $\Omega=\{s:(i,j),\ 1 \le i,j \le M\}$ and we are interested in fitting a Gaussian MRF model to this data. We also assume that the specific structure of the MRF model characterized by the neighbor set N is known. The problem of estimating N has been considered elsewhere [1,8]. Three estimation methods, namely the coding method [7], the maximum likelihood (ML) method, and the method in [9] can be used to obtain estimates of parameters characterizing a MRF model. It is the intent of this paper to analyze in some detail the statistical properties of the estimate in [1,8].

Specifically, we shall derive an expression for the asymptotic mean square error of the estimate for an arbitrary MRF model. We evaluate this expression for a simple first order isotropic MRF model and compare the efficiency of the estimate with the popular coding estimate.

2. Estimates and their statistical properties in MRF models

2.1 Model Representation

Assume that the given image data $\{y(s)\}$ obeys the MRF model in (2.1), with the associated neighbor set N:

$$y(s)=\sum_{r\in N}\theta_r y(s+r)+e(s),\ s\in\Omega, \tag{2.1}$$

In (2.1) the neighbor set N is symmetric, i.e., if $r\in N$ then $-r\in N$. Further, the coefficients satisfy the constraint $\theta_r=\theta_{-r}$. The stationary noise sequence $\{e(s)\}$ is partly characterized by

$$E(e(s)|y(r))=0 \qquad \forall s\neq r \tag{2.2}$$

Using (2.1) and (2.2) one can prove that the noise sequence $\{e(s)\}$ is correlated with the correlation structure

$$\begin{aligned} E(e(s)\ e(r)) &= \nu, & s=r \\ &= -\theta_{s-r}\nu, & (s-r)\in N \\ &= 0 & \text{otherwise} \end{aligned} \tag{2.3}$$

It can be shown by methods similar to [10,11] that an observation y(s) obeying (2.1) with the conditions (2.3) indeed satisfies the Markov condition

$$p(y(s)|y(r),r\in\Omega,r\neq s) = p(y(s)|y(s+t),t\in N)$$

We shall characterize the neighbor set for the MRF model using the set N_S, which includes members from the non-symmetric half of N, i.e., if $s\in N_S$ then $-s\in N_S$ and $N=\{s:s\in N_S\}\cup\{-s:s\in N_S\}$. A sufficient condition to ensure stationarity is [7]

$$\sum\sum\theta_{k,\ell}z_1^k z_2^\ell < 1 \text{ whenever } |z_1| = |z_2| = 1$$

2.2 A consistent estimation scheme [1,8]:

Consider the estimates

$$\underset{\sim}{\theta}^* = [\sum_{\Omega_I} \underset{\sim}{q}(s)\, \underset{\sim}{q}^T(s)]^{-1} (\sum_{\Omega_I} \underset{\sim}{q}(s)\, y(s)) \qquad (2.4)$$

and

$$\nu^* = \frac{1}{M^2} \sum_{\Omega_I} (y(s) - \underset{\sim}{\theta}^{*T} \underset{\sim}{q}(s))^2 \qquad (2.5)$$

where

$$\Omega_I = \Omega - \Omega_B$$

and

$$\Omega_B = \{s = (i,j) : s \in \Omega \text{ and } (s+r) \notin \Omega \text{ for at least one } r \in N\}$$

We now state a theorem regarding the consistency of the estimate $\underset{\sim}{\theta}^*$ and give an expression for the asymptotic variance of the estimate $\underset{\sim}{\theta}^*$. An expression for the asymptotic variance of $\underset{\sim}{\theta}^*$ for an isotropic MRF model with $N_S = \{(0,1),(1,0)\}$ is also given.

Theorem 1: Let $y(s)$, $s \in \Omega$ be the set of observations obeying the MRF model (2.1). Then

(i) The estimate $\underset{\sim}{\theta}^*$ is asymptotically consistent.

(ii) The asymptotic covariance matrix of $\underset{\sim}{\theta}^*$ is

$$E(\underset{\sim}{\theta} - \underset{\sim}{\theta}^*)(\underset{\sim}{\theta} - \underset{\sim}{\theta}^*)^T = \frac{1}{M^2} [\nu \underset{\sim}{Q}^{-1} + 2\nu^2 (\underset{\sim}{Q}^T \underset{\sim}{Q})^{-1} - \frac{\nu}{M^2} \underset{\sim}{Q}^{-1} \sum_{s} \sum_{\substack{r \\ (s-r) \in N}} \theta(s-r) \underset{\sim}{T}_{r,s} (\underset{\sim}{Q}^T)^{-1}] \qquad (2.6)$$

where

$$\underset{\sim}{Q} = E[\underset{\sim}{q}(s) \underset{\sim}{q}^T(s)]$$

and

$$\underset{\sim}{T}_{r,s} = E[\underset{\sim}{q}(r) \underset{\sim}{q}^T(s)]$$

(iii) For the isotropic conditional model with $N_S = \{(0,1),(1,0)\}$, the asymptotic expected mean square error is

$$E(\theta - \theta^*)^2 = \frac{2\theta^2(1-4\theta\alpha_{1,0})^2}{4M^2\alpha_{1,0}^2} \tag{2.7}$$

$$\alpha_{1,0} = \frac{\mathrm{cov}(y(s),y(s+(1,0)))}{\mathrm{cov}(y^2(s))} \tag{2.8}$$

The elements of matrices $\underset{\sim}{Q}$ and $\underset{\sim}{T}_{r,s}$ are functions of normalized autocorrelation coefficients $\alpha_{k,\ell}$. The proof is given in Appendix I.

Although $\underset{\sim}{\theta}^*$ is a consistent estimate of $\underset{\sim}{\theta}$, it is not very efficient. We compare the efficiency of this estimate with efficiencies of the coding estimate for a simple MRF model with $N_S = \{(0,1),(1,0)\}$. The exact ML estimate is obtained by assuming a toroidal lattice representation for y(s) and maximizing the resulting log likelihood function by using the Newton-Raphson procedure. Equation (2.4) can also be used for toroidal lattice representation by summing over Ω instead of Ω_I. The resulting difference in the numerical value is negligible for sufficiently large M.

2.3 Comparison of estimates

We compare the asymptotic variance of the estimate (2.4) with the asymptotic variances of the coding estimate and ML estimate for the isotropic conditional models with $N_S = \{(0,1),(1,0)\}$. From [9], the asymptotic variance of the coding estimate θ_C is

$$M^2 \mathrm{Var}(\theta_C) = \frac{\theta(1-4\theta\alpha_{1,0})}{2\alpha_{1,0}} \tag{2.9}$$

Also from [9], the variance of the ML estimate θ_{ML} is

$$\mathrm{Var}(\theta_{ML}) = \frac{0.5}{M^2(I(\theta) - 4V_{10}^2(\theta))} \tag{2.10}$$

where

$$I(\theta) = \frac{1}{4\pi^2} \iint_0^{2\pi} \frac{(\cos x+\cos y)^2 dxdy}{(1-2\theta(\cos x+\cos y))^2}$$

and

$$v_{st}(\theta) = \frac{1}{4\pi^2} \iint_0^{2\pi} \frac{\cos(sx+ty)dxdy}{(1-2\theta(\cos x+\cos y))}$$

Tabulated values of $V_{10}(\theta)$, $\alpha_{1,0}$ and $I(\theta)$ are available in [9] for different values of θ. Using these values, and (2.7), (2.9) and (2.10), columns 2-4 of Table I are computed. The asymptotic efficiencies in columns 5 and 6 are defined by

$$\mathrm{eff}(\theta_C) = \frac{\mathrm{Var}(\hat{\theta}_{ML})}{\mathrm{Var}(\theta_C)}$$

and

$$\mathrm{eff}(\theta^*) = \frac{\mathrm{Var}(\hat{\theta}_{ML})}{\mathrm{Var}(\theta^*)}$$

It is evident that the estimate $\underset{\sim}{\theta}^*$ computed using (2.4) is more efficient than the coding estimate but is not as good as the ML estimate. Note that column 5 is available in [9].

Appendix I

Proof of Theorem 1:

i) We have from (2.4)

$$\underset{\sim}{\theta}^* = [\sum_{\Omega_I} \underset{\sim}{q}(s)\underset{\sim}{q}^T(s)]^{-1}(\sum_{\Omega_I} \underset{\sim}{q}(s)y(s)) \tag{1}$$

Substituting for y(s) from (2.1) and simplifying, we have

$$[\sum_{\Omega_I} \underset{\sim}{q}(s)\underset{\sim}{q}^T(s)](\underset{\sim}{\theta}^*-\underset{\sim}{\theta}) = \sum_{\Omega_I} \underset{\sim}{q}(s)e(s) \tag{2}$$

Since $E(\underset{\sim}{q}(s)e(s)) = 0$ by (2.2) and $[\sum \underset{\sim}{q}(s)\underset{\sim}{q}^T(s)]$ is a positive definite matrix, the consistency of the estimate $\underset{\sim}{\theta}^*$ follows.

[•]

ii) To make our calculations easy we assume from now on that e(s) is normally distributed. Multiplying (2) by its transpose and taking expectations, we have

$$E[\sum_s \underset{\sim}{q}(s)\underset{\sim}{q}^T(s)(\underset{\sim}{\theta}^*-\underset{\sim}{\theta})(\underset{\sim}{\theta}^*-\underset{\sim}{\theta})^T(\sum_s \underset{\sim}{q}(s)\underset{\sim}{q}^T(s))^T]$$

$$= E[\sum_s \underset{\sim}{q}(s)e(s)\sum_r (\underset{\sim}{q}(r)e(r))^T] \tag{3}$$

$$=\sum_s\sum_r E(\underset{\sim}{q}(s)e(s))\; E(\underset{\sim}{q}^T(r)\underset{\sim}{e}(r))$$

$$+\sum_s\sum_r E(\underset{\sim}{q}(s)e(r))\; E(\underset{\sim}{q}^T(r)e(s))$$

$$+\sum_s \sum_r E(e(s)e(r))\ E(\underset{\sim}{q}(r)\underset{\sim}{q}^T(s)) \tag{4}$$

$$= I + II + III, \text{ where} \tag{5}$$

$$I = 0, \quad \text{by using } (2.2); \tag{6}$$

$$II = 2M^2 \nu^2 I_{mxm}, \text{ by using } (2.2); \tag{7}$$

$$III = \nu \sum_s E(\underset{\sim}{q}(s)\underset{\sim}{q}^T(s))$$

$$- \nu \sum_{\substack{r\ s \\ (s-r)\epsilon N}}\sum \theta_{(s-r)}\ E(\underset{\sim}{q}(r)\underset{\sim}{q}^T(s)) \tag{8}$$

Defining

$$E(\underset{\sim}{q}(s)\underset{\sim}{q}^T(s)) = \underset{\sim}{Q}, \quad \text{mxm matrix}$$

$$E(\underset{\sim}{q}(r)\underset{\sim}{q}^T(s)) = \underset{\sim}{T}_{r,s}, \quad \text{mxm matrix}$$

$$III = M^2 \nu \underset{\sim}{Q} - \nu \sum_{\substack{s\ r \\ (s-r)\epsilon N}}\sum \theta_{(s-r)}\ \underset{\sim}{T}_{r,s} \tag{9}$$

Substituting (6), (7), and (9), we see that

RHS of (3)

$$= M^2[\nu \underset{\sim}{Q} + 2\nu^2 I_{mxm} - \nu \sum_{\substack{s\ r \\ (s-r)\epsilon N}}\sum \theta_{(s-r)}\ \underset{\sim}{T}_{r,s}] \tag{10}$$

for large values of M

$$\frac{1}{M^2}\sum_s \underset{\sim}{q}(s)\underset{\sim}{q}^T(s) = \underset{\sim}{Q} + \eta(M), \tag{11}$$

where $\eta(M)$ is such that

$$E(\eta^2(M)) = O\left(\frac{1}{M^2}\right)$$

Using (11) we obtain

$(\text{LHS of } (3))/M^4 =$

$$E[(\underset{\sim}{Q} + \eta(M))(\underset{\sim}{\theta}^* - \underset{\sim}{\theta})(\underset{\sim}{\theta}^* - \underset{\sim}{\theta})^T(\underset{\sim}{Q} + \eta(M))^T] \tag{12}$$

$$= \underset{\sim}{Q}\, E(\underset{\sim}{\theta}^* - \underset{\sim}{\theta})(\underset{\sim}{\theta}^* - \underset{\sim}{\theta})^T\, \underset{\sim}{Q}^T + O(1/M^2), \tag{13}$$

Substitution of (10) and (13) into (3) yields

$$E(\underset{\sim}{\theta}^* - \underset{\sim}{\theta})(\underset{\sim}{\theta}^* - \underset{\sim}{\theta})^T = \frac{1}{M^2}[\nu\, \underset{\sim}{Q}^{-1} + 2\nu^2(\underset{\sim}{Q}^2)^{-1}$$

$$- \frac{\nu}{M^2}\underset{\sim}{Q}^{-1} \sum_s \sum_{\substack{r \\ (s-r)\in N}} \theta_{(s-r)}\, \underset{\sim}{T}_{r,s}(\underset{\sim}{Q})^{-1} + O(1/M^4) \tag{14}$$

Q.E.D.

c) For the isotropic conditional model with $N_S = \{(0,1),(1,0)\}$, (14) reduces to

$$E(\theta^* - \theta)^2 = \frac{1}{M^2}\frac{1}{(E(q^2(s))^2}[4\nu^2$$

$$+ \nu E(q^2(s)) - \theta\nu \sum_{s}\sum_{r}{}_{(s-r)\epsilon N} E(q(s)q(r))] \qquad (15)$$

where

$$q(s) = y(s+(0,1)) + y(s+(0,-1))$$

$$+ y(s+(1,0)) + y(s+(-1,0)) \qquad (16)$$

Let

$$\gamma_{k,l} = E[y(s)y(s+(k,l))] \qquad (17)$$

Note

$$\gamma_{k,l} = \gamma_{-k,-l} \text{ and } \gamma_{l,k} = \gamma_{k,l} \qquad (18)$$

Express the higher order correlations $\gamma_{2,1}$, $\gamma_{1,-2}$, $\gamma_{2,0}$ and $\gamma_{3,0}$ in terms of $\gamma_{0,0}$, $\gamma_{0,1}$, $\gamma_{1,0}$ and $\gamma_{1,1}$ by

$$\gamma_{0,0} = \nu/(1-4\theta\alpha_{1,0}) \qquad (19)$$

$$\gamma_{2,1} = \frac{1}{2\theta}\gamma_{1,1} - \gamma_{1,0} \qquad (20)$$

$$\gamma_{3,0} = \gamma_{1,0}\{1 + \frac{1}{\theta^2}\} - \frac{\gamma_{0,0}}{\theta} - \frac{3\gamma_{1,1}}{2\theta} - \frac{3\gamma_{1,-1}}{2\theta}$$

$$(21)$$

$$\gamma_{2,0} = \frac{1}{\theta}\gamma_{1,0} - \gamma_{1,-1} - \gamma_{0,0} - \gamma_{1,1} \tag{22}$$

$$\gamma_{1,-2} = \frac{1}{2\theta}\gamma_{1,-1} - \gamma_{1,0} \tag{23}$$

Equations (19-23) can be obtained by multiplying y(s) by appropriately shifted y(s+(k,l)) and taking expectations.

Consider the various terms in (15).

$$E(q^2(s)) = \frac{1}{\theta}4\gamma_{1,0} \tag{24}$$

$$\sum_{s}\sum_{\substack{r \\ (s-r)\epsilon N}} E(q(s)q(r)) = 4[9\gamma_{1,0} + \gamma_{3,0} + 3\gamma_{2,1} + 3\gamma_{1,-2}]$$

$$= 4[4\gamma_{1,0} + \frac{1}{\theta^2}\gamma_{1,0} - \frac{1}{\theta}\gamma_{0,0}] \tag{25}$$

Substitution of (24) and (25) in (15) yields

$$E(\theta^*-\theta)^2 = \frac{1}{M^2}\frac{\theta^2}{16\gamma_{1,0}^2}[4\nu^2 + 4\nu\gamma_{0,0} - 16\theta\gamma_{1,0}\nu]$$

which on using (19) and $\alpha_{1,0} = \gamma_{1,0}/\gamma_{1,1}$ gives

$$E(\theta^*-\theta)^2 = \frac{2\theta^2(1-4\theta\alpha_{1,0})^2}{4M^2\alpha_{1,0}^2}$$

Table 1.* Computation of Asymptotic Variances and Efficiencies of Different Estimates in Isotropic Conditional Model with $N_S = \{(0,1),(1,0)\}$

4θ	$M^2 var(\hat{\theta}_{ML})$	$M^2 var(\theta_C)$	$M^2 var(\theta^*)$	$eff(\theta_C)$	$eff(\theta^*)$
.1	.4928	.497	.494	.991	.9975
.2	.472	.489	.478	.965	.987
.3	.437	.474	.450	.921	.971
.4	.390	.454	.412	.859	.946
.5	.333	.427	.365	.779	.912
.6	.267	.393	.309	.681	.864
.7	.197	.349	.244	.564	.807
.8	.1243	.296	.1753	.419	.709
.9	.0556	.224	.1004	.248	.553

*Column 5 is from [9].

References

1) R. Chellappa, "Fitting Markov random field models to images," Technical Report TR-994, Computer Science Center, University of Maryland, College Park, Maryland, 20742, Jan. 1981.

2) M. Hassner and J. Sklansky, "The use of Markov random fields as models of textures," Computer Graphics Image Proc., Vol. 12, pp. 357-370, April 1980.

3) G.R. Cross, "Markov random field texture models," Technical Report No. 80-02, Department of Computer Science, Michigan State University, East Lansing, Michigan, 1980.

4) R.L. Kashyap and R. Chellappa, "Image restoration using random fields," Proc. of the Eighteenth Annual Allerton Conf. on Comm., Control and Computing, University of Illinois, Urbana, Ill., pp. 856-965, Oct. 1980.

5) J.W. Woods, "Markov image modeling," IEEE Trans. Automat. Contr., Vol. AC-23, pp. 846-850, Oct. 1978.

6) A.K. Jain, "A fast Karhunen-Loeve transform for digital restoration of images degraded by white and colored noise," IEEE Trans. Computers, Vol. C-26, pp. 560-571, June 1977.

7) J.E. Besag, "Spatial interaction and statistical analysis of lattice systems," J. Royal Stat. Soc., Ser. B, Vol. B-36, pp. 192-236, 1974.

8) R.L. Kashyap and R. Chellappa, "Estimation and choice of neighbors in spatial interaction models of images" (submitted for publication).

9) P.A.P. Moran and J.E. Besag, "On the estimation and testing of spatial interaction in Gaussian lattices," Biometrika, Vol. 62, pp. 555-562, 1975.

10) J.W. Woods, "Two-dimensional discrete Markov random fields," IEEE Trans. Information Theory, Vol. 18, pp. 232-240, March 1972.

11) R.L. Kashyap, "Random field models on finite lattices for finite images" (presented at the Symposium on Information Sciences, Baltimore, MD, March , 1981).

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE | READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: **AFOSR-TR- 81 -0556**
2. GOVT ACCESSION NO.: AD-A102
3. RECIPIENT'S CATALOG NUMBER: 057
4. TITLE (and Subtitle): ON AN ESTIMATION SCHEME FOR GAUSS MARKOV RANDOM FIELD MODELS
5. TYPE OF REPORT & PERIOD COVERED: Technical
6. PERFORMING ORG. REPORT NUMBER: TR-1040
7. AUTHOR(s): R. Chellappa
8. CONTRACT OR GRANT NUMBER(s): AFOSR-77-3271
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Computer Vision Laboratory, Computer Science Center, University of Maryland, College Park, MD 20742
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A2
11. CONTROLLING OFFICE NAME AND ADDRESS: Math. & Info. Sciences, AFOSR/NM, Bolling AFB, Washington, DC 20332
12. REPORT DATE: April 1981
13. NUMBER OF PAGES: 15
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
Image processing
Image models
Random fields
Estimation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In an earlier report [1] a consistent estimation scheme was given for Gaussian Markov random field models. In this report we conside some statistical properties of the resulting estimate. Specifically, we derive an expression for the asymptotic mean square error of the estimate for a general model and compare the efficiency of this estimate with the popular coding estimate for a simple first order isotropic model.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DAT
FILM